Math 4200-001
Monday August 31: 1 Section 1.3, complex transformations.

Announcements

On Friday we showed that every non-zero complex number has two square roots. I meant to tell you right after that, and before we talked about the general fundamental theorem of algebra, that a consequence of the square root analysis is that the quadratic formula for any degree 2 quadratic equation

$$a z^2 + b z + c = 0, \; a, b, c \in \mathbb{C}, \; a \neq 0$$
$$z = -\frac{b \pm \sqrt{b^2 - 4 a c}}{2 a}$$

always yields two roots, counting multiplicity. (The $\pm\sqrt{b^2 - 4 a c}$ represents the two square roots of the discriminant, when it's non-zero.)

Warm-up exercise

On Friday we began discussing complex transformations f from $\mathbb{C} \rightarrow \mathbb{C}$. Using polar form we saw that the affine transformations

Example 1

$$f(z) = a\,z + b$$

are compositions of (1) a rotation, followed by (2) a scaling, followed by (3) a translation: Writing

$$z = |\,z\,|\,\mathrm{e}^{i\,arg(z)},$$
$$a = |a\,|\,\mathrm{e}^{i\,arg(a)},$$

we wrote

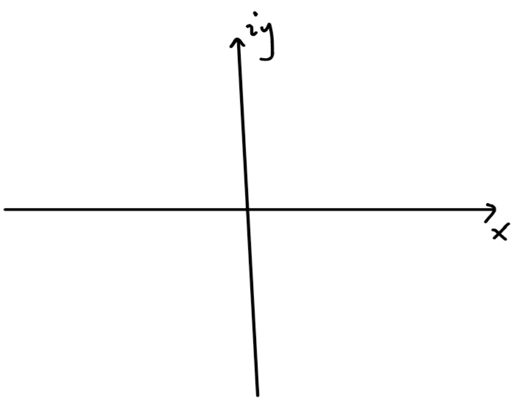
$$f(z) = |a\,|\mathrm{e}^{i\,arg\,a}\,z + b =$$
$$f = f_3 \circ f_2 \circ f_1$$
$$f_1(z) = \mathrm{e}^{i\,arg(a)} z \qquad \text{rotate}$$
$$f_2(w) = |a|\,w \qquad \text{scale}$$
$$f_3(q) = q + b \qquad \text{translate.}$$

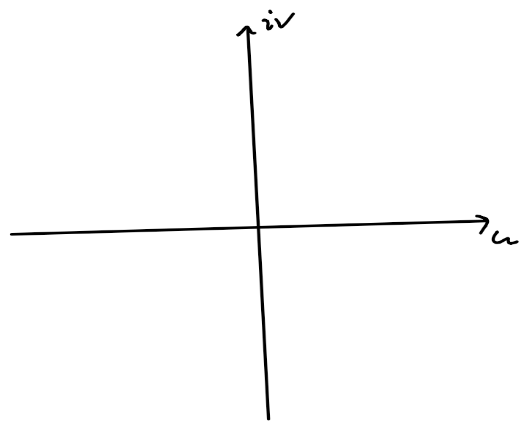
And we saw that it was possible to describe the corresponding transformation $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and the equivalent decomposition using real-variables rotations, scalings, and translations.

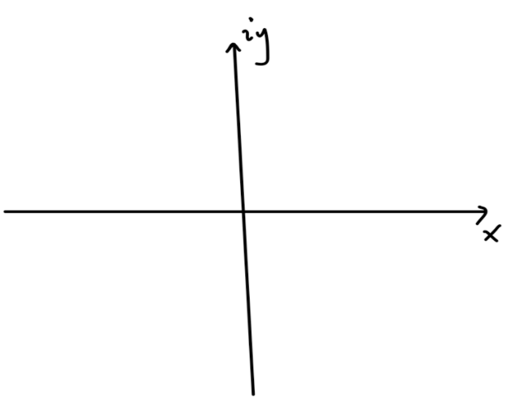
Example 2

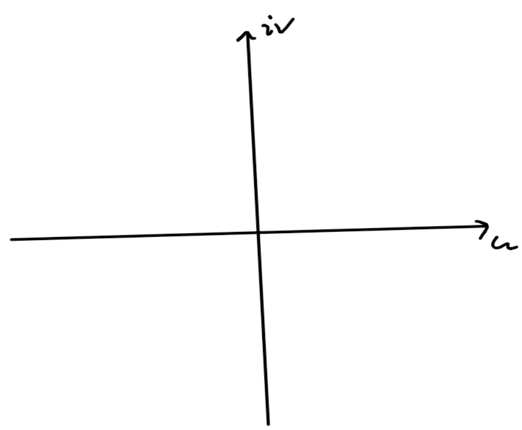
$$f(z) = z^2$$
$$z = |z| \, e^{i \, arg(z)}$$
$$f(z) = |z|^2 \, e^{i \, 2 \, arg(z)}$$

Discuss and sketch how f transforms $z-$plane into a (mostly) twice-covered $w-$plane. For $w = z^2$ discuss possible continuous inverse functions $z = \sqrt{w}$, and corresponding open connected domains are are almost all of $\mathbb{C}$.





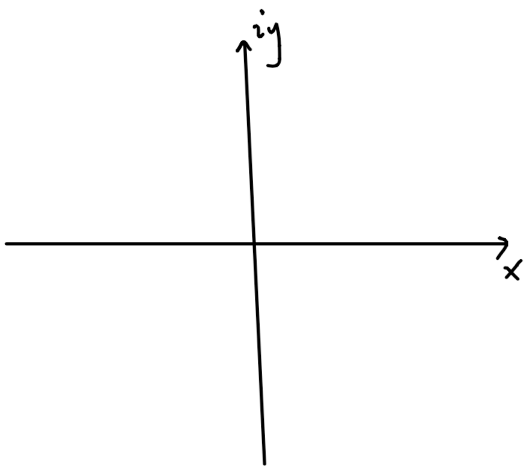


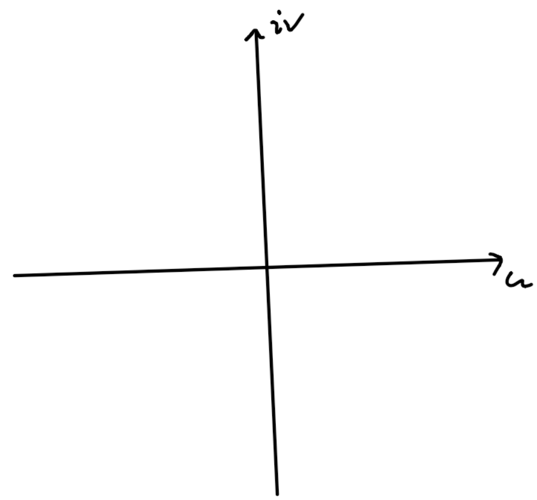


Example 3 For $z = x + i\,y$, $x, y \in \mathbb{R}$

$$f(z) = e^{z} = e^{x + i\,y} := e^{x}\, e^{i\,y}$$

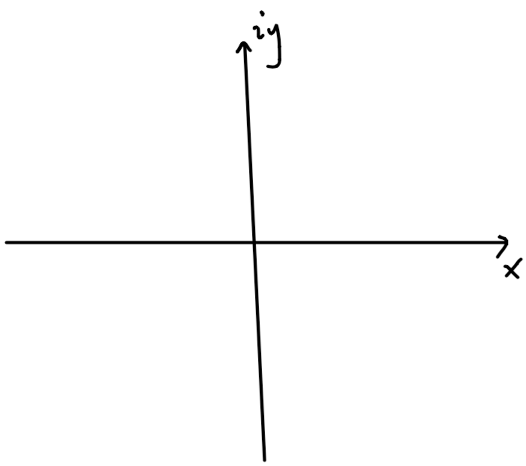
Discuss and sketch how f transforms the $z-$plane into an infinitely covered $w-$plane.

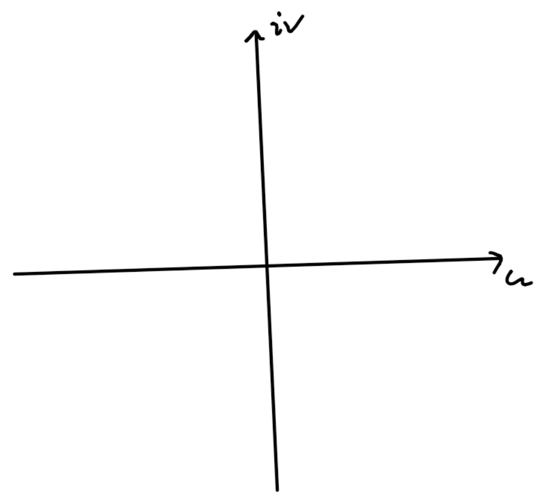




Example 4 For $w = e^z$ use polar form to find the multi-valued inverse "function" $z = \log(w)$, and corresponding domains:

$$e^{x + i y} = w = | w | e^{i \, arg(w)}$$





Remark: The logarithm is used to define complex powers of complex numbers, in analogy with the definition in real variables. It is not too hard to check that this definition generalizes the integer powers and roots that we've already talked about.

Definition:

$$w^z := e^{z \log w}.$$

(I assigned a few hw problems which are examples of this definition.)

Example 5 "Trig functions".
If x is real,

$$\text{eqtn 1} \qquad e^{ix} = \cos(x) + i\sin(x)$$
$$\text{eqtn 2} \qquad e^{-ix} = \cos(x) - i\sin(x).$$

$$\frac{\textit{eqtn } 1 + \textit{eqtn2}}{2} \Rightarrow$$

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\frac{\textit{eqtn } 1 - \textit{eqtn2}}{2\,i} \Rightarrow$$

$$\sin(x) = \frac{1}{2\,i}\left(e^{ix} - e^{-ix}\right).$$

Also recall the hyperbolic trig functions

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{x}\right)$$
$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right).$$

So we define, for $z \in \mathbb{C}$,

$$\cos(z) := \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \cosh(z) := \frac{1}{2}\left(e^{z} + e^{-z}\right) = \cos(i\,z)$$
$$\sin(z) = \frac{1}{2\,i}\left(e^{iz} - e^{-iz}\right) \qquad \sinh(z) := \frac{1}{2}\left(e^{z} - e^{-z}\right) = -i\sin(i\,z).$$

"Trig" identities hold, via properties of complex exponential multiplication. Note that $\sin(z)$, $\cos(z)$ are no longer bounded functions....and it's pretty challenging to figure out their transformation pictures, and their multi-valued inverse functions!

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

$$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$

trigh ...

$$\cosh^2(z) - \sinh^2(z) = \cos^2(iz) + \sin^2(i\,z) = 1 \quad .....$$